PARTIAL FRACTION DECOMPOSITION

USE: rational functions when degree of numerator < degree of denominator.
If degree of numerator $\geq$ degree of denominator, do polynomial division first.

A. DISTINCT LINEAR FACTORS: example: $\dfrac{5x-2}{x^3-4x}$

1. Factor and separate denominator. Numerators are variables representing constants, denominators are linear expressions (factors of original denominator).

ex. $$\frac{5x-2}{x^3-4x}=\frac{A}{x}+\frac{B}{x+2}+\frac{C}{x-2}$$

2. Multiply both sides by least common denominator (LCD).

ex. $$5x-2=A(x+2)(x-2)+Bx(x-2)+Cx(x+2)$$

3. Function has domain of $\Re - \{0,-2,2\}$ so substitute those values in for x.

ex. for $x=0$ equation in step 2 becomes $-2=-4A$ so $A=\frac{1}{2}$
for $x=-2$ equation in step 2 becomes $-12=8B$ so $B=\frac{-3}{2}$
for $x=2$ equation in step 2 becomes $8=8C$ so $C=1$

4. Substitute A, B, and C into equation in step 1 to get:

ex. $$\frac{5x-2}{x^3-4x}=\frac{\frac{1}{2}}{x}+\frac{\frac{-3}{2}}{x+2}+\frac{1}{x-2}=\frac{1}{2x}-\frac{3}{2x+4}+\frac{1}{x-2}$$

B. REPEATED LINEAR FACTORS: example: $\dfrac{2x}{(x-1)^3}$

1. Decompose denominator into progression of linear factor(s) with increasing exponents up to original exponent. Numerators are variables representing constants.

ex. $$\frac{2x}{(x-1)^3}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{(x-1)^3}$$

2. Multiply both sides by LCD. ex. $2x=A(x-1)^2+B(x-1)+C$

3. Original equation has domain of $\Re - \{1\}$ so substitute 1 in for x.

ex. for $x=1$ equation in step 2 becomes $2=C$

4. Substitute any other numbers you wish into equation of step 2 to get equations involving A and B. Use easy numbers like ±1, 0, etc. Result will be two equations with two variables. Solve these equations simultaneously to get values for A and B.

ex. for $x=-1$ and $C=2$, equation in step 2 becomes $-2=4A-2B+2$

ex. for $x=0$ and $C=2$, equation in step 2 becomes $0=A-B+2$

Solving these two equations simultaneously gives $A=0$ and $B=2$

5. Substitute A, B, and C into equation in step 1 to get:

ex. $$\frac{2x}{(x-1)^3} = \frac{0}{x-1} + \frac{2}{(x-1)^2} + \frac{2}{(x-1)^3} = \frac{2}{(x-1)^2} + \frac{2}{(x-1)^3}$$

C. DISTINCT LINEAR AND QUADRATIC FACTORS: example: $\dfrac{x^2+3x-1}{(x+1)(x^2-2)}$

1. Decompose denominators by factoring. Linear denominators have constant numerators, quadratic denominators have linear numerators.

ex. $$\frac{x^2+3x-1}{(x+1)(x^2-2)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-2}$$

2. Multiply both sides by LCD. ex. $x^2+3x-1 = A(x^2-2)+(Bx+C)(x+1)$

3. Substitute integer values not in domain

ex. for $x=-1$ equation in step 2 becomes $3=A$

4. Substitute any other numbers you wish into equation of step 2 to get values for B and C. Use easy numbers like ±1, 0, etc.

ex. for $x=0$ and ***A* = 3**, equation in step 2 becomes ***C* = 5**

ex. for $x=1$, ***A* = 3**, and ***C* = 5**, equation in step 2 becomes **B = -2**

5. Substitute A, B, and C into equation in step 1 to get:

ex. $$\frac{x^2+3x-1}{(x+1)(x^2-2)} = \frac{3}{x+1} + \frac{-2x+5}{x^2-2}$$

D. ALTERNATIVE METHOD FOR LINEAR AND QUADRATIC FACTORS

Do #1 and #2 as above (Linear and Quadratic Factors)

3. Multiply out equation from step #2 and collect like terms:

$$x^2+3x-1 = Ax^2-2A+Bx^2+Bx+Cx+C$$

equate coefficients of like powers on left and right

$$1x^2+3x-1 = (A+B)x^2+(B+C)x+(C-2A)$$

4. so $\mathbf{i: 1 = A+B,\ ii: 3 = B+C,\ iii: -1 = C-2A}$

Solve as simultaneous equations to find $A=3,\ B=-2,\ C=5$

5. Substitute A, B, and C in equation from step 1 above.